NASA TECHNICAL NOTE

NASA TN D-3808

NASA TN D-3808

CONCERNING THE PROPER ROTATIONS OF A 3-DIMENSIONAL ORTHOGONAL FRAME

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • FEBRUARY 1967

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ABSTRACT

For the case of the 3! multiplications of the matrices M_x, M_y, M_z expressing the proper rotations of a three-dimensional orthogonal frame, two special propositions are demonstrated. One is that the trace functions $\psi_- \sim \{M_x M_y M_z,\ M_y M_z M_x,\ M_z M_x M_y\}$ and $\psi_+ \sim \{M_x M_z M_y,\ M_y M_x M_z,\ M_z M_y M_x\}$ are each invariant with respect to the choice of a right or left handed coordinate system. The other proposition is that changes in sense of an odd number of rotation angles about the coordinate axes cause the trace function ψ_- to map onto the trace function ψ_+ or vice versa.

Also, three-dimensional eigendyadics corresponding to elements of the symmetric group $\mathcal{S}_3$ from specified elements of the group $O^+(3)$, and the independence of the choice of an eigendyadic from the order arrangement of the dyadic components is demonstrated.

Thus, the initial choice of spatial orientation of the three-dimensional eigenbasis is immaterial; but once it is made the remaining possible five orientations resulting from the permutational arrangements of the components with respect to the basis elements are uniquely determined.

CONTENTS

Abstract ii

INTRODUCTION 1

FIRST PROPOSITION 2

SECOND PROPOSITION 3

CONSTRUCTION OF 3-DIMENSIONAL EIGENDYADICS CORRESPONDING TO ELEMENTS OF THE SYMMETRIC GROUP $\mathcal{S}_3$ 3

Bibliography 6

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INTRODUCTION

This treatise was conceived with a view toward various problems of spatial orientation in classical mechanics, optics, and electromagnetic field theory. It is a kinematical study of proper rotations occurring in three-dimensional space. A result is obtained in which the elements of the symmetric group on three letters are related to eigendyadics. These eigendyadics have as their components eigenvalues which are derived from the 3! different rotation operators considered.

The ensuing discussion is a study of the six different elements of the group of proper rotations, the proper orthogonal group $O^+(3)$ (or as it is otherwise called, the special orthogonal group $SO(3)$), which are formed as products of the three elements M_x, M_y, and M_z, expressing rotation about the three coordinate axes of the orthogonal frame ${}_3\mathfrak{J}$. These 3! products are formed by permuting the order of the multiplication of M_x, M_y, and M_z. The rotation angles shall be so chosen that no one of the unimodular factors M_x, M_y, and/or M_z is an identity element: thus $\theta_i \neq \pm 2k\pi$ where $i = 1, 2,$ or 3 and $k = 0$ or a positive integer; for in that case, the system reduces to one involving only two proper rotation matrices. Such a case would then be a constraint placed upon the system as originally conceived, which would affect the generality of the conclusions. The angles θ_1, θ_2, and θ_3 of rotation about the x, y, and z axes, are elements of the field $R^{\#}$ of real numbers. ${}_3\mathfrak{J}$ itself is embedded in a three-dimensional continuum which, though it could be globally Euclidean, is at the least locally Euclidean.

The explicit representations of the three proper rotation matrices are:

$$M_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}, \quad M_y = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}, \quad M_z = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1}$$

These unimodular matrices may of course, be multiplied together in six different order arrangements to form six different products.

After computation of the corresponding traces of the matrix group elements, it is found that

$$\cos\theta_2\cos\theta_3+\cos\theta_1\cos\theta_2+\cos\theta_1\cos\theta_3-\sin\theta_1\sin\theta_2\sin\theta_3$$

$$=\psi_- = \mathrm{tr}\,(M_x M_y M_z) = \mathrm{tr}\,(M_y M_z M_x) = \mathrm{tr}\,(M_z M_x M_y); \qquad (2)$$

and

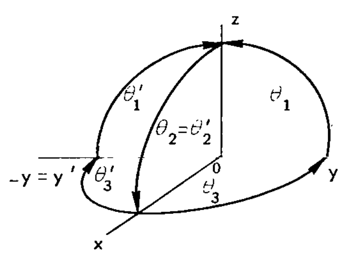
$$\cos\theta_2\cos\theta_3+\cos\theta_1\cos\theta_2+\cos\theta_1\cos\theta_3+\sin\theta_1\sin\theta_2\sin\theta_3$$

$$=\psi_+ = \mathrm{tr}\,(M_x M_z M_y) = \mathrm{tr}\,(M_y M_x M_z) = \mathrm{tr}\,(M_z M_y M_x). \qquad (3)$$

The permutational arrangement of the group products $M_x M_y M_z$, $M_y M_z M_x$, and $M_z M_x M_y$ of 0^+ (3) corresponding to the trace ψ_- corresponds to a cyclic permutation of letters. Similarly the cyclic arrangement $M_x M_z M_y$, $M_y M_x M_z$, and $M_z M_y M_x$ corresponds to the trace ψ_+. The array of cyclic permutations corresponding to ψ_+ may be achieved by an initial inversion of two letters from the array corresponding to ψ_-.

FIRST PROPOSITION

The trace functions ψ_- and ψ_+ are each invariant with respect to a choice of a right or left handed coordinate system.



Consider the diagram where the primed angle refers to a left handed system, and where $\theta_3' = -\theta_3$, $\theta_1' = -\theta_1$, and $\theta_2' = \theta_2$.

With respect to ψ_-, consider the term $-\sin\theta_1\sin\theta_2\sin\theta_3$. By substituting the left handed angle elements into this term, it is readily seen that

$$-\sin\theta_1\sin\theta_2\sin\theta_3 = -\sin\theta_1'\sin\theta_2'\sin\theta_3'.$$

Similarly, with respect to ψ_+, substituting the left handed angle elements into the term $+\sin\theta_1\sin\theta_2\sin\theta_3$ shows that

$$+\sin\theta_1\sin\theta_2\sin\theta_3 = \sin\theta_1'\sin\theta_2'\sin\theta_3'.$$

Thus the choice of the sign preceding the term $\sin\theta_1 \sin\theta_2 \sin\theta_3$ (the sign which distinguishes ψ_- from ψ_+) is independent of a choice of a right or left-handed coordinate system. Q.E.D.

SECOND PROPOSITION

By changing at least one angle, or at most three different angle elements (but not two) from a positive to a negative sense, the trace function ψ_- will be altered to become ψ_+, or the trace function ψ_+ can be mapped onto ψ_-.

When $\theta_i = \pm 2k\pi$, $i = 1, 2, 3$ where $k = 0$, a positive integer, or a positive rational fraction, then $\psi_- \equiv \psi_+$ for at least one of the angles of rotation. This case is equivalent to the condition that at least one of the rotation matrices of the set $\{M_x, M_y, M_z\}$ corresponds to the identity matrix I_3. By definition, a rotation of 180° represents a reflection and corresponds to $k = \pm(2n+1)/2$ where $n = 0$ or a positive integer: or $k = \pm q/2$ where $q \equiv 0 \bmod 1$, $q = 0$ or a positive integer.

Consider the trace function ψ_- (Relation 2):

Case I: If θ_1 is replaced by $-\theta_1$, θ_2 by $-\theta_2$, and θ_3 by $-\theta_3$, then $\psi_- = \psi_+$.

Case II: If θ_1 is replaced by $-\theta_1$ and θ_2 by $-\theta_2$, or if θ_1 is replaced by $-\theta_1$ and θ_3 by $-\theta_3$, or if θ_2 is replaced by $-\theta_2$ and θ_3 by $-\theta_3$, then $\psi_- = \psi_-$.

Case III: If θ_1 is replaced by $-\theta_1$, or if θ_2 is replaced by $-\theta_2$, or if θ_3 is replaced by $-\theta_3$, then $\psi_- = \psi_+$.

Thus, it is seen that two cases arise, where i is the number of angle element changes which are altered in sense; they are cases I and III, and are expressed by the relation $i \not\equiv 0 \bmod 2$. This means that if one angle is replaced by its negative, or if all 3 angles are replaced by their negatives, then $\psi_- \to \psi_+$ or $\psi_+ \to \psi_-$. Case II is expressed by the relation $i \equiv 0 \bmod 2$. This means that if two angles are replaced by their negatives, then $\psi_- \to \psi_-$ and $\psi_+ \to \psi_+$. Q.E.D.

CONSTRUCTION OF 3-DIMENSIONAL EIGENDYADICS CORRESPONDING TO ELEMENTS OF THE SYMMETRIC GROUP $\mathcal{S}_3$

The two sets of proper rotation matrices

$$\{M_x M_y M_z, M_y M_z M_x, M_z M_x M_y\} \sim \lambda^3 - \psi_- \lambda^2 + \psi_- \lambda - 1 = 0 \tag{4}$$

and

$$\{M_y M_x M_z, M_x M_z M_y, M_z M_y M_x\} \sim \lambda^3 - \psi_+ \lambda^2 + \psi_+ \lambda - 1 = 0 \tag{5}$$

where

$$\lambda^3 - \psi_\pm \lambda^2 + \psi_\pm \lambda - 1 = 0 ,$$

are the characteristic equations of the matrices to which they correspond. The constant term of the characteristic equation is the determinant of the product of the rotation matrices. It is therefore +1 in value because of the unimodularity of the group elements of $O^+(3)$, and so the constant term and its sign is $(-1)^3 \cdot 1$. The eigenvalues of these proper rotation operators, which are the roots of the characteristic equations (Relations 4 and 5) may be arranged as follows.

Corresponding to ψ_-:

$$\lambda_1^- = +1 ,$$
$$\lambda_2^- = \frac{\psi_- - 1 + \sqrt{(1-\psi_-)^2 - 4}}{2} = e^{i \cos^{-1}(\psi_- - 1)/2} = e^{i \sin^{-1}(3 + 2\psi_- - \psi_-^2)^{1/2}/2} ,$$
$$\lambda_3^- = \frac{\psi_- - 1 - \sqrt{(1-\psi_-)^2 - 4}}{2} = e^{-i \cos^{-1}(\psi_- - 1)/2} = e^{-i \sin^{-1}(3 + 2\psi_- - \psi_-^2)^{1/2}/2} .$$

And corresponding to ψ_+:

$$\lambda_1^+ = +1 ,$$
$$\lambda_2^+ = \frac{\psi_+ - 1 + \sqrt{(1-\psi_+)^2 - 4}}{2} = e^{i \cos^{-1}(\psi_+ - 1)/2} = e^{i \sin^{-1}(3 + 2\psi_+ - \psi_+^2)^{1/2}/2} ,$$
$$\lambda_3^+ = \frac{\psi_+ - 1 - \sqrt{(1-\psi_+)^2 - 4}}{2} = e^{-i \cos^{-1}(\psi_+ - 1)/2} = e^{-i \sin^{-1}(3 + 2\psi_+ - \psi_+^2)^{1/2}/2} .$$

It is easy to calculate that $x' = (\psi_\pm - 1)/2$ and $y' = (3 + 2\psi_\pm - \psi_\pm^2)^{1/2}/2$, and that the eigenvalues lie on the unit circle in the complex plane. For conciseness of notation, let $\cos^{-1}(\psi_- - 1)/2 = \alpha$ and $\cos^{-1}(\psi_+ - 1)/2 = \beta$. The eigenvalues may now be represented as

$$\lambda_1^- = 1 , \qquad \lambda_1^+ = 1 ,$$
$$\lambda_2^- = e^{i\alpha} , \qquad \lambda_2^+ = e^{i\beta} ,$$
$$\lambda_3^- = e^{-i\alpha} , \qquad \lambda_3^+ = e^{-i\beta} .$$

It may be of interest here to note that the trace functions ψ_- and ψ_+ may also be expressed as $\psi_- = 1 + e^{i\alpha} + e^{-i\alpha} = 1 + 2\cos\alpha$; and $\psi_+ = 1 + e^{i\beta} + e^{-i\beta} = 1 + 2\cos\beta$.

The two inequivalent sets of matrices $\{M_x M_y M_z, M_y M_z M_x, M_z M_x M_y\}$ and $\{M_y M_x M_z, M_x M_z M_y, M_z M_y M_x\}$ have been shown to be convertible to each other by changing the sense of an odd number of angles for all the elements of any one set. And this will have the effect of initially permuting two of the elements of any one of the matrix products contained in one of the sets of matrices, and then cyclically permuting the new order arrangement. In this manner, the order of the factors of any product of one set can be made identical to the order of a corresponding product of matrices in the other set. Thus, an equivalence has been effected by this conversion transformation. These transformations may otherwise be expressed as

$$\begin{aligned}\psi_{\pm}(\theta_1, \theta_2, \theta_3) &= \psi_{\mp}(-\theta_1, \theta_2, \theta_3)\\ &= \psi_{\mp}(\theta_1, -\theta_2, \theta_3)\\ &= \psi_{\mp}(\theta_1, \theta_2, -\theta_3)\\ &= \psi_{\mp}(-\theta_1, -\theta_2, -\theta_3).\end{aligned}$$

Thus the eigenvalues $e^{\pm i\alpha}$ and $e^{\pm i\beta}$ are indistinguishable with respect to such a transformation, and since $e^{\pm i\alpha} = e^{\pm i\beta} \Longrightarrow \alpha \equiv \beta$, there is no theoretical distinction between these two sets of eigenvalues.

Because of the commutativity of the linear factors of the characteristic polynomial, which will now be written as $\lambda^3 - \psi\lambda^2 + \psi\lambda - 1$, it is immaterial which index, 1, 2, or 3, actually designates which root expression. In addition, the existence of 6 different choices for the order positioning of the eigenvalues is also evident. These order arrangements can be considered as diagonal elements of a dyadic in nonion form, and can be expressed as:

$$\begin{aligned}{}^2_3\mathbf{T}_1 &= \mathbf{ii} + e^{i\alpha}\,\mathbf{jj} + e^{-i\alpha}\,\mathbf{kk}\ ,\\ {}^2_3\mathbf{T}_{(123)} &= e^{i\alpha}\,\mathbf{ii} + e^{-i\alpha}\,\mathbf{jj} + \mathbf{kk}\ ,\\ {}^2_3\mathbf{T}_{(132)} &= e^{-i\alpha}\,\mathbf{ii} + \mathbf{jj} + e^{i\alpha}\,\mathbf{kk}\ ,\\ {}^2_3\mathbf{T}_{(12)} &= e^{i\alpha}\,\mathbf{ii} + \mathbf{jj} + e^{-i\alpha}\,\mathbf{kk}\ ,\\ {}^2_3\mathbf{T}_{(23)} &= \mathbf{ii} + e^{-i\alpha}\,\mathbf{jj} + e^{i\alpha}\,\mathbf{kk}\ ,\\ {}^2_3\mathbf{T}_{(13)} &= e^{-i\alpha}\,\mathbf{ii} + e^{i\alpha}\,\mathbf{jj} + \mathbf{kk}\ ,\end{aligned}$$

where the superscript preceding the tensor symbol is the order of the tensor (in this case 2), the subscript preceding the tensor symbol is the dimension of the space, and the subscript following the tensor symbol represents the particular order arrangement of the dyadic components by referring to the permutation element of the symmetric group on 3 letters, $\mathcal{S}_3$. The alternating group $\mathcal{A}_3 \cong \{1, (123), (132); \cdot\}$ is the normal subgroup of $\mathcal{S}_3$, and corresponds to those tensors whose components have been cyclically permuted with respect to the basis elements (the unit dyads **ii**, **jj**, and **kk**); whereas the permutations (12), (23), and (13), correspond to those tensors whose components have been cyclically permuted after an initial inversion of the dyadic corresponding to the arrangement chosen to correspond to the identity permutation.

Choose the set of three dyadics which, by virtue of the cyclical permutation of their components with respect to their bases dyad elements **ii**, **jj**, and **kk**, correspond to the elements 1,(123), and (132) of the alternating group $\mathfrak{A}_3$. This choice is accomplished by making an initial choice from one of the 3! possible arrangements and calling it the identity. These dyadics have been established as ${}^2_3\mathbf{T}_1$, ${}^2_3\mathbf{T}_{(123)}$, and ${}^2_3\mathbf{T}_{(132)}$. Or, if it is wished, choose that set of three dyadics which was constructed upon a choice of an inversion of two of the components of the dyadic corresponding to the identity permutation, and whose components were then twice cyclically permuted, ${}^2_3\mathbf{T}_{(12)}$, ${}^2_3\mathbf{T}_{(23)}$, and ${}^2_3\mathbf{T}_{(13)}$.

Consider the set of dyadics $\left\{{}^2_3\mathbf{T}_{(12)}, {}^2_3\mathbf{T}_{(23)}, {}^2_3\mathbf{T}_{(13)}\right\}$. If the angle elements α and $-\alpha$ of their components $e^{i\alpha}$ and $e^{-i\alpha}$ are exchanged, which is in effect a replacement of them by substitution, the following transformation α_p occurs:

$$\alpha_p : {}^2_3\mathbf{T}_{(12)} \rightarrow {}^2_3\mathbf{T}_{(132)} ,$$

$$\alpha_p : {}^2_3\mathbf{T}_{(23)} \rightarrow {}^2_3\mathbf{T}_1 ,$$

$$\alpha_p : {}^2_3\mathbf{T}_{(13)} \rightarrow {}^2_3\mathbf{T}_{(123)} .$$

The choice of order of the components 1, $e^{i\alpha}$, and $e^{-i\alpha}$, with respect to the unit dyads **ii**, **jj**, and **kk**, which form the 3 dyadics corresponding to the permutations 1, (123), and (132), is immaterial. It is possible to choose any one of the three dyadics ${}^2_3\mathbf{T}_1$, ${}^2_3\mathbf{T}_{(123)}$, and ${}^2_3\mathbf{T}_{(132)}$ to represent a unique set of basis elements for an eigendyadic.

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NASA-Langley, 1967

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